The tangent complex

Want this "Ψ^{-1}"

Recap:

$$\begin{tikzcd}
\mathrm{Lie}_k \arrow[rr, "\Psi"] \arrow[dr, "C^*_{\mathrm{Lie}}"] \arrow[ddr, bend right, "\text{forget}"'] & & \mathrm{FMP} \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S}) \arrow[ll, bend right] \arrow[ddl, dashed, "? =: F"] \\
& (\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}} \arrow[ur, "\mathrm{Spf}"] \arrow[ul, dotted, bend left, "\mathcal{D}"] & \\
& \mathrm{Mod}_k &
\end{tikzcd}$$

Will constructs this "Koszul duality functor"

Goal: Construct the Koszul duality functor $\mathcal{D}$

Baby step: Let's linearize the problem – find $? =: F$ with controlling property $F(\Psi(\mathfrak{g})) \simeq \mathfrak{g}$ as dg v.sp.

Hint: $\Psi(\mathfrak{g})(A) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}}(C^*_{\mathrm{Lie}}(\mathfrak{g}), A)$

$\rightsquigarrow$ How can we linearize the algebra $C^*_{\mathrm{Lie}}(\mathfrak{g})$?

Guess: Take the Zariski tangent space of the point

$$\mathrm{Spec}\, k \xrightarrow{\mathrm{Spec}(\varepsilon)} \mathrm{Spec}(C^*\mathfrak{g}), \text{ where}$$

$\varepsilon : C^*_{\mathrm{Lie}}\mathfrak{g} \longrightarrow k \cong \mathrm{Sym}^0$ is the augmentation

$T^*_{\mathrm{Zar}} = \mathfrak{m}_\varepsilon / \mathfrak{m}_\varepsilon^2$ Zariski cotangent space

$T_{\mathrm{Zar}} = \mathrm{Hom}(\mathfrak{m}_\varepsilon/\mathfrak{m}_\varepsilon^2, k)$ Zariski tangent space

What is $\mathfrak{m}_\varepsilon$ in this case?

$$\varepsilon : C^*_{\mathrm{Lie}}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}^*[-1]) \circlearrowleft^{d} \longrightarrow \mathrm{Sym}^0(\ ) \cong k$$

$$\mathfrak{m}_\varepsilon = \ker \varepsilon = \mathrm{Sym}^{\geq 1}(\mathfrak{g}^*[-1]) \supseteq (\mathfrak{g}^*[-1])$$

$\Longrightarrow \quad {}^{m_\varepsilon}/_{m_\varepsilon^2} = \mathfrak{g}^*[-1] \Longrightarrow \boxed{T_{Zar} = \mathfrak{g}[1]}$

<u>Side remark:</u> I'm glad you're worried whether this is "homotopically correct".

<u>Fact:</u>
$$\begin{tikzcd} A \arrow[dr] & & C \arrow[dl] \\ & B & \end{tikzcd}$$
if A, B, C are fibrant & at least one arrow is a fibration, then the "naive" pullback $A \times_B C$ is a homotopy pullback.

in dgVect:
$$\begin{tikzcd} C^*(\mathfrak{g}) \arrow[dr, two heads, "\varepsilon"'] & & 0 \arrow[dl] \\ & k & \end{tikzcd}$$
fibration $\Longrightarrow$ naive pullback $= m_\varepsilon$ is a homotopy pullback

Upshot: the naive tangent space is homotopically correct in this example.

<u>Goal for today:</u> Find a version of the "tangent space" of a formal moduli problem X!

$\rightsquigarrow$ Need to rephrase tangent space so that we don't use an algebra, e.g. $C^*_{Lie}(\mathfrak{g})$.

<u>Functor of points:</u> $A \xrightarrow{\varepsilon} k \quad \longleftrightarrow \quad \operatorname{Spec} k \xrightarrow{p_\varepsilon} \operatorname{Spec} A$

$$T_p = \left\{ \begin{tikzcd} \operatorname{Spec} k \arrow[r, "p"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} k[x]/(x^2) \arrow[r] \arrow[ur, dashed, "\tilde{p}"] & \operatorname{Spec} k \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} k & A \arrow[l, "1"'] \arrow[d, "\varepsilon"] \arrow[dl, dashed, "\tilde{\varepsilon}"'] \\ k[x]/(x^2) \arrow[u, "1"'] \arrow[r] & k \end{tikzcd} \right\}$$

$|x| = 0$

This definition agrees with the previous one:

$$\begin{tikzcd}
\mathfrak{m}_\varepsilon^2 \arrow[d] \arrow[dr, bend left, "0 \;\swarrow\; x^2=0"] & \\
\mathfrak{m}_\varepsilon \arrow[r, "\text{suffices}"', "\text{to give this}"] \arrow[d] & \mathfrak{m} = k\cdot x \arrow[d] \\
A \arrow[r, dashed, "\tilde{\varepsilon}"] \arrow[d] & k[x]/(x^2) \arrow[d] \\
k \arrow[r, "\mathrm{id}"'] & k
\end{tikzcd}$$

A map $\tilde{\varepsilon}$ is determined by a k-linear map $\mathfrak{m}_\varepsilon \to k\cdot x$ vanishing on $\mathfrak{m}_\varepsilon^2 \subset \mathfrak{m}_\varepsilon$

$\Rightarrow T_p \cong \mathrm{Hom}_k(\mathfrak{m}_e/\mathfrak{m}_e^2, k)$

$\Longrightarrow$ Can use the lifting problem for arbitrary functors on CAlg!

<u>Basic idea</u> for generalizing to the derived situation:

Replace the dual numbers by <u>all</u> of the shifted dual numbers

$$k[x_{-d}]/(x_{-d}^2), \quad |x_{-d}| = -d \quad d \geq 0$$

Then to $X \in \mathrm{FMP}$, we get a collection of spaces

$$\left\{ X\left(k[x_{-d}]/(x_{-d}^2) \right) \right\}_{d \in \mathbb{N}}$$

This collection forms the homotopical analog of a vector space, namely, a

$$\underbrace{k\text{-module}}_{\substack{\text{analog of being}\\ \text{a module over } k}} \quad \underbrace{\text{spectrum}}_{\text{homotopy analog of abelian gp}}$$

<u>Fact</u>: There is an equivalence of ∞-categories

$$\left\{ \begin{matrix} k\text{-module} \\ \text{spectra} \end{matrix} \right\} \simeq \mathrm{dgVect}_k$$

Spectra

Def'n Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces,

$\mathcal{S}_*^{fin}$ the smallest full ∞-subcategory containing the terminal object $* \longrightarrow *$ that is closed under finite colimits.

Remark: 1) One way to construct $\mathcal{S}_*^{fin}$ is to consider

$$sSet_*^{fin} = \{ * \longrightarrow X : X \text{ has finitely many non-degenerate simplices} \}$$

2) From a topological perspective, you can use finite cell complexes w/ a base point.

3) "Spheres" generate $\mathcal{S}_*^{fin}$ under pushouts.

Def'n: Let $\mathcal{C}$ be a pointed ∞-category, i.e. $\mathcal{C}$ has both an initial and a terminal object and these are equivalent, denoted $0_{\mathcal{C}}$.

A <u>spectrum object</u> in $\mathcal{C}$ is a functor $F : \mathcal{S}_*^{fin} \longrightarrow \mathcal{C}$

s.t. 1) $F(*) \simeq 0_{\mathcal{C}}$ <u>"reduced"</u>

2) F sends a pushout square to a pullback square. <u>"excisive"</u>

Prop: Suppose $\mathcal{C}$ possesses finite colimits. A spectrum $F : \mathcal{S}_*^{fin} \to \mathcal{C}$ is determined by a sequence $F_E = \{E_n\}_{n \in \mathbb{N}}$ w/ $E_n \in \mathcal{C}$ and equivalences

$$E_n \simeq \Omega E_{n+1} := 0_{\mathcal{C}} \times_{E_{n+1}} 0_{\mathcal{C}}$$

Proof: Given F, define $E_n := F(S^n)$.

Recall: $S^{n+1} \simeq \Sigma S^n := * \coprod_{S^n} *$

Remember:

$$\begin{tikzcd} & S^n \arrow[dl] \arrow[dr] & \\ * \arrow[dr] & h & * \arrow[dl] \\ & \, & \end{tikzcd} \quad \overset{\text{compute}}{\underset{\text{as}}{\rightsquigarrow}} \quad \begin{tikzcd} & S^n \arrow[dl] \arrow[dr] & \\ * & & D^n \end{tikzcd}$$

naive pushout is S^{n+1}

F is excisive, so get pullback

$$\begin{tikzcd} F(S^n) =: E_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & E_{n+1} := F(S^{n+1}) \end{tikzcd} \implies E_n \simeq \Omega E_{n+1}$$

Conversely, given such a sequence & equivalences observe that every finite cell complex is obtained by a sequence of pushouts using spheres. Hence F is determined (up to a contractible space of choices) ∎

<u>Remark</u>: The classic example of a spectrum is a <u>cohomology theory</u>

$$H^* : S_*^{fin} \longrightarrow Ch_{\mathbb{Z}},$$

where $H^*(*) = 0$ & H^* satisfies excision.

<u>Prop</u>: The sequence of shifted dual numbers $\{k \oplus k\varepsilon_{-d}\}_{d \in \mathbb{N}}$ is a spectrum in $CAlg^{aug}$, denoted by "Dual".

<u>Pf</u>: By the previous proposition, we only need to show that

$$k \oplus k\varepsilon_{-d} \simeq \Omega(k \oplus k\varepsilon_{-d-1}) = \underbrace{k \underset{k \oplus k\varepsilon_{-d-1}}{\times} k}_{\infty\text{-categorical pullback}}$$

Let's compute this explicitly by computing the homotopy pullback in the model cat. $Ch_{\mathbb{Z}}$

$$\begin{tikzcd}
k \simeq k \oplus \underbrace{(k\varepsilon_{-d-1} \xrightarrow{\text{"id"}} k\varepsilon_{-d})}_{\text{cone(id)}} \arrow[d, "\text{id}"', bend right] \arrow[d, "0", bend left] \\
k \arrow[r] & k \oplus k\varepsilon_{-d-1}
\end{tikzcd}$$

acyclic cplx

fibration!

Now can compute naive pullback of $\longrightarrow \swarrow$

$$= k \oplus k\varepsilon_{-d}$$ ∎

<u>Lemma</u>: This spectrum factors through $CAlg^{sm}$! i.e.,

$$\begin{tikzcd}
S_*^{fin} \arrow[r, "\text{Dual}"] \arrow[rr, bend right, "\text{Dual}"'] & CAlg^{sm} \arrow[r, hook] & CAlg^{aug}
\end{tikzcd}.$$

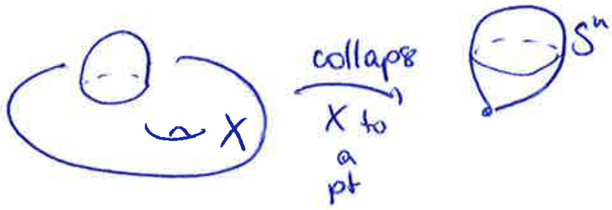
$K \to K'$ in st. surjective on π_0

Proof: Let ~~K~~ $\mathcal{S}^{fin}_*$. Then there is a sequence ~~of "cell attachments"~~

$$K = K_0 \to K_1 \to \cdots \to K_n = K',$$

where each map $K_i \to K_{i+1}$ sits in a pushout square

$$\begin{tikzcd} K_i \arrow[r] \arrow[d] & * \arrow[d] \\ K_{i+1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^n \end{tikzcd}$$



Then the spectrum Dual sends this to a pullback

$$\begin{tikzcd} \mathrm{Dual}(K_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Dual}(*) \simeq k \arrow[d] \\ \mathrm{Dual}(K_{i+1}) \arrow[r] & \mathrm{Dual}(S^n) \simeq k \oplus k\varepsilon_{-n} \end{tikzcd}$$

Hence the map

$$\mathrm{Dual}(K_i) \to \mathrm{Dual}(K_{i+1}) \text{ is elementary,}$$

$$\text{so } \mathrm{Dual}(K) \to \mathrm{Dual}(K') \text{ is small.}$$

$\blacksquare$

Def'n: For $X \in \mathrm{FMP}$, the composite

$$X \circ \mathrm{Dual} : \mathcal{S}^{fin}_* \to \mathcal{S}$$

is a spectrum object, called the <u>tangent complex of X</u>, denoted by $\mathbb{T}_X$.

(uses 2)

Lemma: $\mathbb{T}_X$ is a spectrum.

Thm Let $f: X \to Y$ be a map of FMP. Then,

f is an equivalence iff $\mathbb{T}_f := f \circ \mathrm{Dual} : \mathbb{T}_X \to \mathbb{T}_Y$ is an equivalence.

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Proof: $\Rightarrow$: obvious

$\Leftarrow$: For $A \in CAlg^{sm}$, pick a sequence of elementary extensions

$$A = A_n \longrightarrow A_{n-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 = k.$$

Use induction on n. $n=0$, then $X(k) \simeq * \simeq Y(k)$

$\Rightarrow f(k)$ is an equivalence.

$n \mapsto n+1$:

$$\begin{tikzcd} A_{i+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k \arrow[d] \\ A_i \arrow[r] & k \oplus k\varepsilon_{-d} \end{tikzcd} \quad \overset{X}{\rightsquigarrow} \quad \begin{tikzcd} X(A_{i+1}) \arrow[r] \arrow[d] & * \arrow[d] \\ X(A_i) \arrow[r] & X(k \oplus k\varepsilon_{-d}) \end{tikzcd}$$

$\overset{Y}{\rightsquigarrow}$ — " —

Applying X and Y, we get a fiber sequence in spaces

$$\begin{tikzcd} X(A_{i+1}) \arrow[r] \arrow[d, "f(A_{i+1})"'] & X(A_i) \arrow[r] \arrow[d, "f(A_i)"', "\simeq \text{ ind hyp.}"] & X(k \oplus k\varepsilon_{-d}) \arrow[d, "f(k\oplus k\varepsilon_{-d})"', "\simeq \text{ by assumption}"] \\ Y(A_{i+1}) \arrow[r] & Y(A_i) \arrow[r] & X(k \oplus k\varepsilon_{-d}) \end{tikzcd}$$

$\Rightarrow f(A_{i+1})$ is an equivalence:

consider the map of long exact sequences of the homotopy gps, since the 2 on the right are equiv., $\pi_* f(A_{i+1})$ are equiv. $\blacksquare$